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Note on Gill's solution for free convection in a vertical enclosure

By ADRIAN BEJAN
Department of Mechanical Engineering, University of Colorado, Boulder

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This paper develops an alternative approach to evaluating the arbitrary constants found in Gill's solution for the boundary-layer free-convection regime in a vertical rectangular enclosure. The new method consists of calculating the net upward flow of energy through the enclosure and setting it equal to zero near the top and bottom boundaries of the cavity. The present method takes into account the impermeable and adiabatic properties of the horizontal end walls. The overall Nusselt number derived on this new basis is shown to agree well with available experimental and numerical heat-transfer correlations.

1. Introduction

Gill (1966) developed a now well-known theory on the boundary-layer regime for free convection in a vertical rectangular cavity with the side walls at different temperatures and the top and bottom end walls adiabatic. This free-convection problem is of great practical importance, particularly because of its pivotal position in the engineering of thermal insulations. In developing his theory, Gill relied heavily on visual observations and experimental measurements reported for the same regime by Elder (1965). The Gill solution was based first on the assumption that a stratified fluid core exists far away from both vertical walls. Boundary-layer solutions were then obtained for the flow near the two vertical walls. Matching the boundary layers with the *same* core solution led to a consistent picture for the free-convection pattern between the two tall vertical walls of the enclosure. However, owing to the boundary-layer-type approximations, this solution cannot satisfy all the boundary conditions physically present in the vertical direction.

In total, four conditions account for the fact that the top and bottom walls are impermeable and adiabatic. Gill's solution contains only two arbitrary constants. As argued by Quon (1977), some judgement is required before deciding which conditions ought to be considered to best determine the two constants. Gill chose to invoke the two impermeability conditions, $\psi = 0$ at $z = \pm\frac{1}{2}$, thus determining one constant as $C = 0{\cdot}912$.† The second constant was found to be zero on the basis of the symmetry of the top and bottom wall conditions imposed, and for this reason it does not appear explicitly in Gill's final solution. Quon (1977) attempted to determine C by matching Gill's solution with a numerical solution for the stream function and vertical velocity profile in the cavity. The difficulty associated with this approach comes from the

† Throughout this article we use the results already derived by Gill in his 1966 paper.

fact that Gill's velocity profile is a discontinuous approximation of the real profile measured by Elder (1965).

The objective of this note is to develop a conceptually different basis for evaluating the constants for Gill's boundary-layer solution. Gill's result is representative of the boundary-layer regime in the cavity, in regions not very close to the horizontal walls at $z = \pm\frac{1}{2}$. Owing to the approximation mentioned above, this result is *expected* to break down in the two regions near the top and bottom boundaries. Therefore it seems inappropriate to pin the boundary-layer solution to conditions present in a domain where the boundary-layer approximation no longer applies. This opinion is also supported by the fact that Gill's theory, with C derived from the impermeable end condition, predicts infinite horizontal velocities u and infinite vertical temperature gradients $\partial T/\partial z$ along the horizontal walls. Moreover, if we try to discard impermeability and use instead the adiabatic condition $\partial T/\partial z = 0$ at $z = \pm\frac{1}{2}$, we soon find that the boundary-layer solution cannot meet an adiabatic condition at any z. Noting that $\partial T/\partial z = 0$ is equivalent to requiring $\partial q/\partial z = 0$, this conclusion is more evident in Quon's 1977 paper, where $q(z)$, which is an odd function of altitude, is displayed graphically.

Consider the physics of the boundary-layer regime in an enclosure where the left wall is warmer than the right wall, as analysed by Gill. Energy is being transferred from left to right somewhat indirectly, by first heating the fluid rising along the left wall. As the fluid circulates clockwise, energy is released as the descending branch comes in contact with the colder wall. The local Nusselt number is highest near the bottom of the left wall and, diametrically opposite, near the top of the right wall. Looking now at the fluid circulating in the enclosure, the natural counterflow carries energy *upwards* between the two vertical walls. This vertical energy 'current' increases from $z = -\frac{1}{2}$ to $z = 0$ as the counterflow receives more energy from the left compared with the energy it looses to the right. The opposite holds in the upper half of the cavity, from $z = 0$ to $z = \frac{1}{2}$. Overall, the vertical energy flux Q per unit time and width is an even function of altitude which reaches its maximum at the midheight $z = 0$. At the same time, the vertical energy current is zero in the top and bottom end regions.

In what follows we use the condition of zero vertical energy flux to determine the arbitrary constant appearing in Gill's solution. This choice is consistent with the physics of the fluid flow inside the cavity, where Gill's approximations apply. At the same time, the statements $Q(\pm\frac{1}{2}) = 0$ take into account *in an average way* all four conditions applicable along the solid top and bottom boundaries, where, as we said earlier, Gill's solution breaks down. There is also a practical reason for modifying the boundary-layer solution along these lines. Correlations for the overall Nusselt number for heat transfer between the vertical walls of the enclosure have been reported in many experimental and numerical investigations, as summarized by Ostrach (1972). It is generally agreed that the overall Nusselt number obeys a relationship of the form $Nu = aRa^b(H/L)^c$ when $2 < H/L < 20$. However, a long-ranging controversy persists regarding the exact values of the 'constants' a, b and c. We demonstrate in this article that a, b and c are actually functions of Ra and H/L. For the first time, a theoretical explanation will be provided for the present disagreement regarding the average Nusselt number Nu.

2. The upward flow of energy

To determine Q, consider an energy-flux integral over any cross-section at constant altitude:

$$Q = \int_0^L \left(\rho c_p w_* T_* - \lambda \frac{\partial T_*}{\partial z_*}\right) dx_*. \tag{1}$$

Here ρ, c_p and λ are the fluid density, specific heat at constant pressure and thermal conductivity, respectively. In terms of Gill's dimensionless variables, (1) becomes

$$\frac{QL}{\lambda H(T_a - T_b)} = \frac{L}{l}\int_0^{L/l} w\theta dx - \frac{Ll}{H^2}\int_0^{L/l} \frac{\partial T}{\partial z} dx. \tag{2}$$

Next we evaluate the integrals accounting for convection and conduction in (2) above. We do this by splitting each integral into two parts corresponding to the two solutions available for the two vertical boundary-layer regions. The procedure involves some algebra, which will be omitted here. The final result is

$$\frac{QL}{\lambda H(T_a - T_b)} = \frac{L}{l}\,\frac{C^3(1-q^2)^3}{4(1+3q^2)^{\frac{11}{3}}} - \left(\frac{L}{H}\right)^2 \frac{2(1+3q^2)^{\frac{53}{6}}}{C^4(7-q^2)\,(1-q^2)^2\,(1+q^2)^{\frac{8}{3}}}, \tag{3}$$

showing that the convective contribution to Q, the first term in (3), increases linearly with L/l, the ratio of cavity thickness to boundary-layer thickness. Recalling that $L/l = (RaL/H)^{\frac{1}{4}}$, the convective part tends to dominate Q as the Rayleigh number Ra increases. Examining result (3), we also find that the conductive contribution to Q decreases if the cavity becomes more slender. Another intuitively obvious feature of (3) is that Q is an even function of z, depending on z only via q^2. The vertical energy current reaches its maximum at the midheight, where $q = 0$.

Zero energy flow at $z = \pm\frac{1}{2}$ then requires

$$C^7 = \left(\frac{H}{L}Ra^{\frac{1}{7}}\right)^{-\frac{7}{4}} \frac{8(1+3q_e^2)^{\frac{75}{6}}}{(7-q_e^2)\,(1-q_e^2)^5\,(1+q_e^2)^{\frac{8}{3}}}, \tag{4}$$

where q_e is the end value of the altitude parameter q, viz.

$$q(\pm\tfrac{1}{2}) = \pm q_e. \tag{5}$$

The end value q_e is actually a function of C which can easily be obtained by integrating Gill's equation (6.19) from $z = 0$ to $z = \frac{1}{2}$. The result of this operation is plotted in figure 1. Combining figure 1 with (4), we now have a means of estimating the boundary-layer-solution constant C (as well as q_e) in terms of the new group $Ra^{\frac{1}{7}}H/L$. The final result provided by the condition of zero energy upflow has been summarized in figure 2. One interesting aspect of this result is that in the limit $Ra^{\frac{1}{7}}H/L \to \infty$ it becomes identical to Gill's result based on top and bottom impermeability, viz. $C = 0{\cdot}912$ and $q_e = 1$. This coincidence is not surprising since in this limit the vertical energy flux is all by convection, hence the requirements of zero energy flux and impermeable end walls become identical.

By considering the $Q = 0$ conditions near the top and bottom of the enclosure, we have shown that Gill's boundary-layer solution is affected not only by the group $Ra\,L/H$ but also by the new group $Ra^{\frac{1}{7}}\,H/L$. That is, when the boundary-layer regime as envisioned by Gill prevails in the cavity, the flow field, temperature pattern and the heat transfer are affected by the Rayleigh number Ra and the aspect ratio H/L

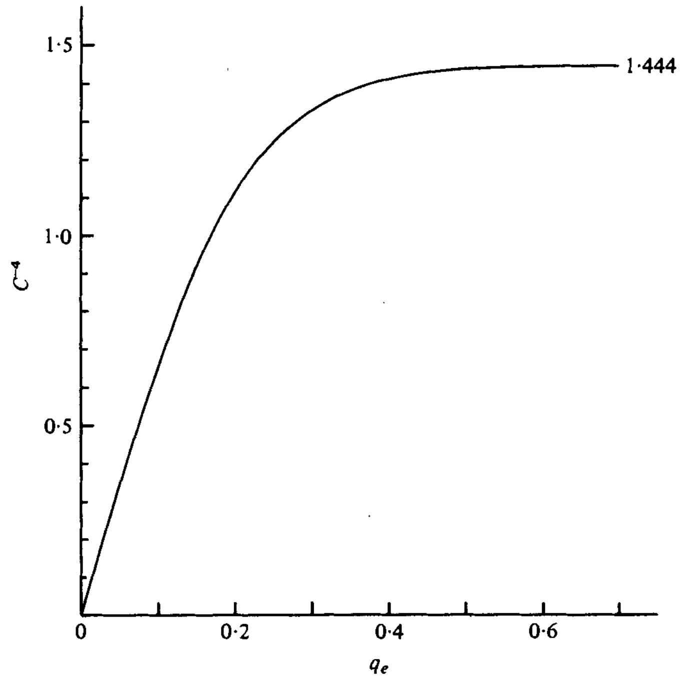

FIGURE 1. Dependence of Gill's constant C on the end parameter $q_e = q(\frac{1}{2})$.

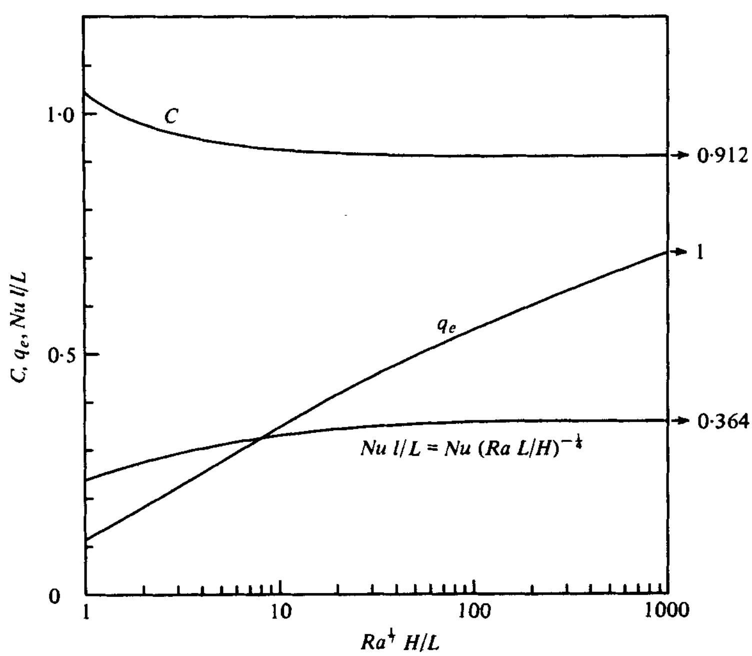

FIGURE 2. Influence of new group $Ra^{\frac{1}{7}}\, H/L$ on C, q_e and $Nu\; l/L$.

independently. As we show below, this conclusion is strongly supported by the Nusselt number data available in the literature. The opportunity of providing for the first time a theoretical explanation for effects observed both experimentally and numerically is the main reason for reporting the present addendum to Gill's solution.

3. The overall Nusselt number

Consider now the overall Nusselt number describing the net heat transfer by free convection between the vertical walls of the enclosure. Gill in his 1966 paper stopped short of evaluating this number, which, as pointed out in the introduction, is a central result sought by many investigators for its practical value. We define the overall Nusselt number as

$$Nu = \frac{WL}{\lambda H(T_a - T_b)} = \frac{L}{l}\int_{-\frac{1}{2}}^{\frac{1}{2}} \left(-\frac{\partial T}{\partial x}\right)_{x=0} dz. \tag{6}$$

Using the boundary-layer solution subject to the condition of zero energy upflow, Nu becomes

$$Nu\left(Ra\frac{L}{H}\right)^{-\frac{1}{4}} = \frac{C^3}{8}\int_{-q_e}^{q_e} \frac{(1-q)^6(1+q)^2(7-q^2)}{(1+q^2)(1+3q^2)^{\frac{14}{3}}}\,dq. \tag{7}$$

The right-hand side of (7) is a number which, via C and q_e (shown in figure 2), depends on the new group $Ra^{\frac{1}{7}}H/L$. The integral appearing in (7) was evaluated numerically and the result is shown as the $Nu(Ra\,L/H)^{-\frac{1}{4}}$ curve on figure 2. In the limit $Ra^{\frac{1}{7}} \times H/L \rightarrow \infty$, which may be regarded here as the 'Gill limit', the overall Nusselt number is given by

$$Nu = 0{\cdot}364\,(Ra\,L/H)^{\frac{1}{4}} \quad \text{as} \quad Ra^{\frac{1}{7}}H/L \rightarrow \infty. \tag{8}$$

The overall Nu given by (8) is only 35 % larger than the local Nu estimated by Gill for the midheight point $z = 0$.

Figure 2 indicates that as soon as the new group $Ra^{\frac{1}{7}}H/L$ is less than about 100 the overall Nusselt number departs from its asymptotic value. In practical cases the departure is of the order of up to 30 %, a change which certainly cannot be overlooked in energy-thrifty thermal insulation systems. Figure 2 can be used directly to evaluate Nu for any given aspect ratio and Rayleigh number based on the spacing between the vertical walls.

4. Discussion

We conclude this article with a brief survey of overall Nu correlations often quoted in the literature (see, for example, Ostrach's 1972 review). Figures 3 and 4 show a limited selection of Nusselt number correlations *vis-à-vis* the theoretical result developed in the preceding section. The experimental correlations displayed on figure 3 were developed by Eckert & Carlson (1961), Jakob (1949), MacGregor & Emery (1968), Seki, Fukusako & Inaba (1978) and Yin, Wung & Chen (1978). The correlations based on numerical solutions of the free-convection heat-transfer problem (figure 4) were reported by De Vahl Davis (see Landis & Rubel 1970), Newell & Schmidt (1970) and Pepper & Harris (1977). The numerical correlations in figure 4 are in superior mutual agreement compared with the experimental results in figure 3. In either case,

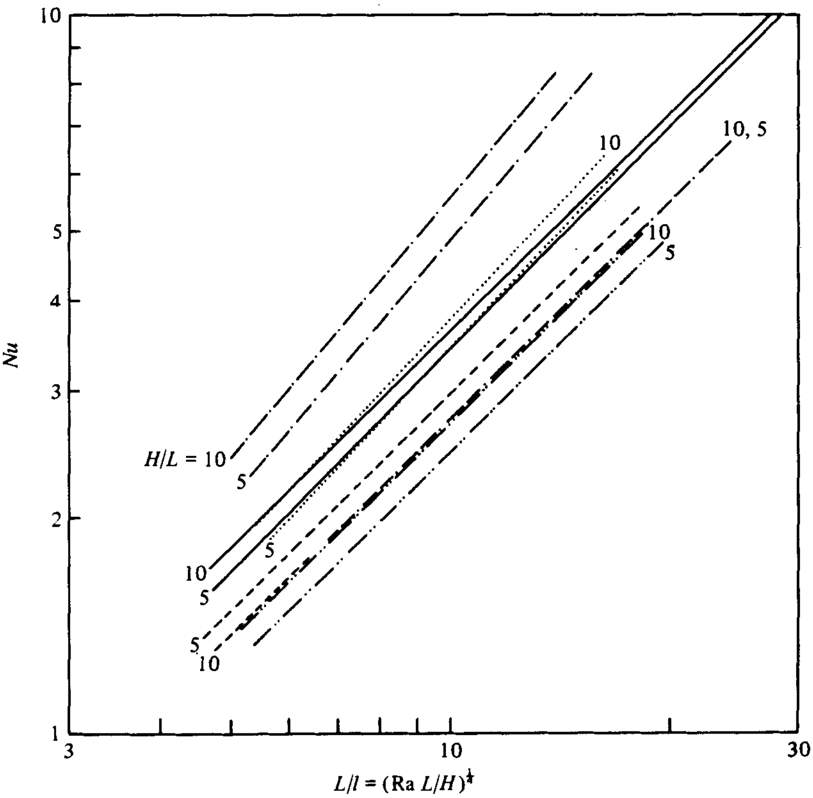

FIGURE 3. Comparison between the present solution and experimental Nusselt number correlations for $H/L = 5$ and 10. ——, present solution; – – –, Seki *et al.* (1978); — · —, Eckert & Carlson (1961); – – · – –, MacGregor & Emery (1968); — · · —, Jakob (1949); · · · · · ·, Yin *et al.* (1978).

the present theory for the overall Nusselt number splits the field covered by these correlations right through the middle.

The agreement between the present theory and the correlations based on numerical simulations is excellent, particularly near $(Ra\, L/H)^{\frac{1}{4}} \approx 10$, which is exactly the range where Gill's boundary-layer model is an acceptable approximation. Below this range of values the heat-transfer mechanism is slowly replaced by direct conduction in the horizontal direction. Above this range the boundary-layer picture becomes considerably more complicated owing to the presence of secondary and tertiary cellular flows, as discussed by Elder (1965).

The theoretical Nusselt number developed in this article can be used with the same if not a higher degree of confidence than can heat-transfer correlations available today. In particular, the Nusselt number corresponding to the Gill limit (equation (8) and curve $H/L \rightarrow \infty$ on figure 4) can be used with confidence for tall cavities, $H/L \gg 10$.

One last observation concerns the end value q_e plotted on figure 2. It is significant that, even when the new group $Ra^{\frac{1}{7}}\, H/L$ is extremely large, q_e differs appreciably from its asymptotic value. The fact that q_e is less than one implies that fluid enters the warm (left) boundary layer at $z = -\frac{1}{2}$ with finite vertical velocity and leaves the same layer at $z = \frac{1}{2}$ with finite vertical velocity also. This means that in the two

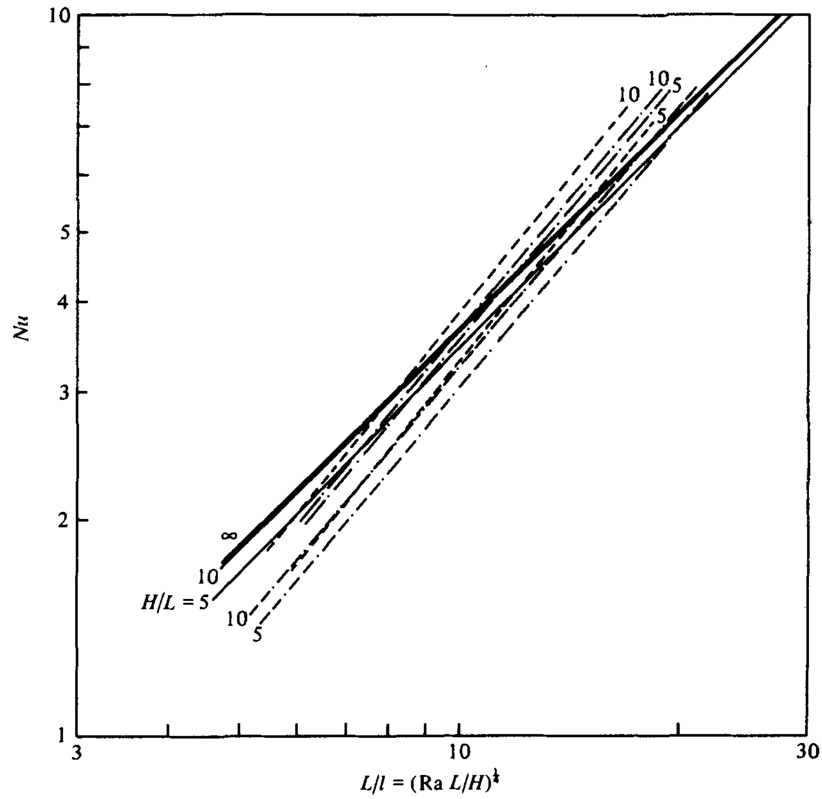

FIGURE 4. Comparison between the present solution and Nusselt number correlations based on numerical studies (H/L = 5 and 10). ——, present solution; – – –, De Vahl Davis (see Landis & Rubel 1970); —·—, Newell & Schmidt; – –·– –, Pepper & Harris (1977).

corners, where the boundary-layer approximation breaks down, the flow field must also contain the following features. In the top corner the fluid flows to the right, hitting the top wall and slightly rebounding back into the cavity. Near the bottom corner a local vacuum draws fluid from the right and slightly above and to the right into the tip of the boundary layer. The present theory implies also that the corner effects will be more pronounced in short cavities, where $Ra^{\frac{1}{4}}\, H/L$ and q_e are small. These observations are in qualitative agreement with published streamline patterns, particularly those of Quon (1972) for square cavities and those of Cormack, Leal & Seinfeld (1974) and Imberger (1974) for shallow rectangular cavities.

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